(1)

(6) A COORDINATISATION OF LATTICES BY ONE-SIDED BAER ASSEMBLIES

(10) T.S. Blyth
William [redacted].C. Hardy

(9) Professional Paper 46

(11) Jul[redacted]70

(12) 30

NOTICE

THIS DOCUMENT HAS BEEN APPROVED FOR PUBLIC RELEASE

(14) CNA-PP-46

CENTER FOR NAVAL ANALYSES
1401 Wilson Boulevard
Arlington, Virginia 22209

077270 gm

A Coördinatisation of Lattices by One-sided Baer Assemblies. By

T.S. Blyth, Mathematical Institute, University of St.Andrews and
W.C. Hardy, Centre for Naval Analyses, Arlington, Virginia. Commu-
nicated by Emeritus Professor E.T. COPSON.

SYNOPSIS

In an earlier publication [1] we introduced the notion of a *Baer assembly* and applied it to obtain a coördinatisation theory for semilattices. This was achieved by considering the semigroup of quasi-residuated (*i.e.*, O-preserving and isotone) mappings on a bounded semilattice. In the present paper we consider the semigroup of quasi-residuated $\cup$-homomorphisms (or *hemimorphisms*) on a bounded lattice and thus show how a particular type of one-sided Baer assembly can be used to provide a coördinatisation theory for lattices; and in particular for complemented, modular and distributive lattices.

1. Introduction.

We have shown in a previous publication [1] how the fundamental coördinatisation of bounded lattices by Baer semigroups can be extended to semilattices through the concept of a Baer assembly. We recall ([1], §3) that $\langle A;B,D;K\rangle$ is a Baer assembly if A is a semigroup, B and D are distinguished subsets of A and K is a two-sided ideal of A such that

$$\begin{cases}(\forall x\in D)(\exists e = e^2\in B) & R_K(x) = eA;\\ (\forall y\in B)(\exists f = f^2\in D) & L_K(y) = Af,\end{cases}$$

where $R_K(x) = \{z\in A;\ xz\in K\}$ and $L_K(y) = \{z\in A;\ zy\in K\}$. In the case where A has a zero element 0 and $K = \{0\}$ we shall agree to drop the suffix K. We shall assume throughout the present paper that the reader is familiar with the terminology and notation used in [1]. Our goal here will be to isolate a type of Baer assembly which will coördinatise a bounded lattice and do so in such a way that the lattice operations are readily recaptured as semigroup conditions. This we then use to provide Baer assembly coördinatisations of complemented, modular and distributive lattices. The latter two results do not as yet have analogues in the theory of Baer semigroups.

As in [1], the results which follow depend heavily on the choice of the semigroup of mappings and the correct choice of idempotents. To establish these, we recall the following definitions and results from [1].

Let E be an ordered set with minimum element 0 and maximum element 1 and let Q denote the semigroup of quasi-residuated (*i.e.*, 0-preserving and isotone) mappings on E. If E is a $\vee$-semilattice we denote by T the subsemigroup of quasi-residuated $\vee$-homomorphisms (or *hemimorphisms*) on E. Finally, we denote by C the subset of Q consisting of those mappings with principal kernels. It is shown in [1] that the mappings θ_x defined by

$$(\forall x \in E) \qquad \theta_x(y) = \begin{cases} y & \text{if } y \leq x; \\ x & \text{if } y \not\leq x, \end{cases}$$

are idempotent elements of T, that the mappings $\hat{\psi}_x$ defined by

$$(\forall x \in E) \qquad \hat{\psi}_x(y) = \begin{cases} 0 & \text{if } y \leq x; \\ x \vee y & \text{if } y \not\leq x, \end{cases}$$

are idempotents in $T \cap C$ and that ([1], §5)

$$\begin{cases} (\forall f \in T \cap C) & R(f) = \theta_{f^+(0)} \circ T; \\ (\forall g \in T) & L(g) = T \circ \hat{\psi}_{g(1)}, \end{cases}$$

where $f^+(0) = \max\{x \in E;\ f(x) = 0\}$. Moreover, if E is a lattice then θ_x is an idempotent in the semigroup $\mathrm{Res}(E)$ of residuated mappings on E.

In this paper we shall restrict our attention to the semigroup T and the mappings θ_x and $\hat{\psi}_x$ will be the idempotents of primary interest in the coördinatisation of lattices.

We recall ([1], §3) that a Baer assembly is said to be *normal* if $K \subseteq B \cap D$ and ([1], §4) that a *right* Baer assembly is a normal Baer assembly $\langle A;B,D;K\rangle$ in which

(1) B is a subsemigroup of A;

(2) $B \subseteq D$;

(3) in the semigroup $P(A)$, $B \subseteq D.{}^{\cdot}D$ [*i.e.*, $DB \subseteq D$].

The notion of a *left* Baer assembly is defined similarly. A right [resp. left] Baer assembly $\langle A;B,D;K\rangle$ is said to *coördinatise* an ordered set E if $E \cong \mathcal{R}(D) = \{R_K(x);\ x \in D\}$ [resp. $E \overset{d}{\cong} \mathcal{L}(B) = \{L_K(x);\ x \in B\}$ where $\overset{d}{\cong}$ denotes dual order isomorphism].

2. Coördinatisation of lattices by one-sided Baer assemblies.

Let $\langle A;B,D;K\rangle$ be a right Baer assembly. We shall use the notation

$$[B] = \{e \in B;\ e = e^2 \text{ and } eA \in \mathcal{R}(D)\};$$

$$[D] = \{f \in D;\ f = f^2 \text{ and } Af \in \mathcal{L}(B)\}.$$

We shall say that $\langle A;B,D;K\rangle$ is a *balanced* right Baer assembly if and only if there exist subsets B_* of $[B]$ and D^* of $[D]$ such that

(1) for each $x \in D$ there is a unique $x_* \in B_*$ such that $R_K(x) = x_*A$;

(2) for each $x \in B$ there is a unique $x^* \in D^*$ such that $L_K(x) = Ax^*$;

(3) $D^* \subseteq B$ [the "balancing factor"];

(4) if $e, f \in B_*$ are such that $e^*f \notin K$ and $f^*e \notin K$ then

$$\begin{cases} L_K(e^*f) = L_K(f^*e) = L_K(e) \cap L_K(f); \\ R_K(e^*f) = R_K(f^*e) = R_K(e^*) \cap R_K(f^*). \end{cases}$$

Remark. As immediate consequences of this definition we have that $L_K(e^*f), L_K(f^*e) \in \mathcal{L}(B)$ and $R_K(e^*f), R_K(f^*e) \in \mathcal{R}(D)$; this follows from the fact that $e^*f, f^*e \in D^*B_* \subseteq BB_* \subseteq B \subseteq D$. Moreover, we have $x \in B_* \iff x = (x^*)_*$ and $x \in D^* \iff x = (x_*)^*$; for example, if $x \in B_*$ then by Theorem 8 of [1] we have $xA = \hat{R}_K[L_K(x)] = \hat{R}_K(Ax^*) = R_K(x^*) = (x^*)_*A$ whence $x = (x^*)_*$ by the uniqueness in (1). Finally, we note at this juncture that condition (4), which expresses conditions on the intersection of right and left annihilators of products of idempotents, can also be expressed as conditions on the elements of B_* and D^*. It can be shown (though we shall not do so here) that, if we define an element $e_* \in B_*$ to be *r-decreasing* whenever

$$(\forall f^* \in D^*)(e_* \notin R_K(f^*)) \qquad R_K(f^*e_*) \subseteq R_K(f^*)$$

and an element $f^* \in D^*$ to be *l-decreasing* whenever

$$(\forall e_* \in B_*)(f^* \notin L_K(e_*)) \qquad L_K(f^*e_*) \subseteq L_K(e_*),$$

then the conditions $L_K(f^*e) = L_K(e) \cap L_K(f)$ and $R_K(f^*e) = R_K(e^*) \cap R_K(f^*)$ are satisfied if and only if each $e_* \in B_*$ is r-decreasing and each $f^* \in D^*$ is l-decreasing. However, we shall use the former conditions in order to facilitate the presentation.

Our basic model of a balanced right Baer assembly is given in the following result.

THEOREM 1 *If E is a bounded lattice then $\langle T; Res(E), T \cap C; \{0\} \rangle$ is a balanced right Baer assembly which coördinatises E.*

Proof. That $\langle T;\mathrm{Res}(E),T\wedge C;\{0\}\rangle$ is a right Baer assembly which coördinatises E is established by observing that the results of §2 of [1] carry over *in toto* to the semigroup T with ψ_y replaced by $\hat{\psi}_y$ so that Theorem 11 of [1] with Q replaced by T, C replaced by $T\wedge C$ and ψ_y replaced by $\hat{\psi}_y$ carries through in exactly the same fashion. To show that this right Baer assembly is balanced, write $\mathrm{Res}(E) = S$ and consider the subsets S_* of $[S]$ and $(T\wedge C)^*$ of $[T\wedge C]$ given by

$$S_* = \{\theta_x;\ x\in E\} \qquad \text{and} \qquad (T\wedge C)^* = \{\hat{\psi}_x;\ x\in E\}.$$

(That each θ_x belongs to $[S]$ follows from Theorem 9 of [1]). Now for each $f\in T\wedge C$ we have $R(f) = \theta_{f^+(0)}\circ T$ where $\theta_{f^+(0)}\in S_*$; and if $\theta_y\in S_*$ is also such that $R(f) = \theta_y\circ T$ then the T-analogue of Theorem 3 of [1] shows that we must have $\theta_y(\pi) = f^+(0)$. But $\theta_y(\pi) = y$ and so $y = f^+(0)$ whence $\theta_y = \theta_{f^+(0)}$. It follows that $\theta_{f^+(0)}$ is the unique element f_* of (1) above. Similarly, for each $f\in S$ we have $L(f) = T\circ\hat{\psi}_{f(\pi)}$ where $\hat{\psi}_{f(\pi)}\in(T\wedge C)^*$; and if $\hat{\psi}_y\in(T\wedge C)^*$ is such that $L(f) = T\circ\hat{\psi}_y$ then the T-analogue of Theorem 4 of [1] gives $y = \hat{\psi}_y^+(0) = f(\pi)$. Thus $\hat{\psi}_{f(\pi)}$ is the unique element f^* of (2) above. To establish (3), we note that, by §5 of [1], each $\hat{\psi}_y\in S$ and so $(T\wedge C)^*\subseteq S$. As for (4), we have, for each $x\in E$,

$$\begin{cases} L(\theta_x) = T\circ\hat{\psi}_{\theta_x(\pi)} = T\circ\hat{\psi}_x\,; \\[2ex] R(\hat{\psi}_x) = \theta_{\hat{\psi}_x^+(0)}\circ T = \theta_x\circ T\,, \end{cases}$$

and so $(\theta_x)^* = \hat{\psi}_x$ and $(\hat{\psi}_x)_* = \theta_x$. Now given $\theta_x,\theta_y\in S_*$ we have

$$[(\theta_x)^*\circ\theta_y](z) = (\hat{\psi}_x\circ\theta_y)(z) = \begin{cases}\hat{\psi}_x(z) & \text{if } z\le y; \\ \hat{\psi}_x(y) & \text{if } z\not\le y,\end{cases}$$

$$= \begin{cases} 0 & \text{if } z \leq y \text{ and } z \leq x; \\ x \cup z & \text{if } z \leq y \text{ and } z \not\leq x; \\ 0 & \text{if } z \not\leq y \text{ and } y \leq x; \\ x \cup y & \text{if } z \not\leq y \text{ and } y \not\leq x. \end{cases}$$

It follows from this that $(\theta_x)^* \circ \theta_y \neq 0$ implies $y \not\leq x$ and so if $(\theta_x)^* \circ \theta_y \neq 0$ and $(\theta_y)^* \circ \theta_x \neq 0$ we have $y \not\leq x$ and $x \not\leq y$, which we express in the usual way by writing $x \| y$. In this case we have

$$(x \| y) \qquad (\hat{\psi}_x \circ \theta_y)(z) = \begin{cases} 0 & \text{if } z \leq x \cap y; \\ x \cup z & \text{if } z \not\leq x \text{ and } z \leq y; \\ x \cup y & \text{if } z \not\leq y, \end{cases}$$

with a similar formula for $(\theta_y)^* \circ \theta_x$. It follows that, in the semi-group T,

$$\begin{aligned} \alpha \in L(\hat{\psi}_x \circ \theta_y) &\iff \alpha(x \cup y) = 0 \iff \alpha \in L(\hat{\psi}_y \circ \theta_x) \\ &\iff \alpha(x) \cup \alpha(y) = 0 \\ &\iff \alpha(x) = 0 = \alpha(y) \\ &\iff \alpha \in L(\theta_x) \cap L(\theta_y), \end{aligned}$$

so that $L(\hat{\psi}_x \circ \theta_y) = L(\hat{\psi}_y \circ \theta_x) = L(\theta_x) \cap L(\theta_y)$. Similarly,

$$\begin{aligned} \alpha \in R(\hat{\psi}_x \circ \theta_x) &\iff \operatorname{Im} \alpha \subseteq [\leftarrow, x \cap y] \iff \alpha \in R(\hat{\psi}_y \circ \theta_x) \\ &\iff \operatorname{Im} \alpha \subseteq [\leftarrow, x] \cap [\leftarrow, y] \\ &\iff \alpha \in R(\hat{\psi}_x) \cap R(\hat{\psi}_y), \end{aligned}$$

so that $R(\hat{\psi}_x \circ \theta_y) = R(\hat{\psi}_y \circ \theta_x) = R(\hat{\psi}_x) \cap R(\hat{\psi}_y)$. This then shows that the right Baer assembly $\langle T; \mathrm{Res}(E), T \cap C; \{0\} \rangle$ is balanced and completes the proof.

The fundamental coördinatisation of bounded lattices by one-sided Baer assemblies may now be stated as follows:

THEOREM 2 *If E is an ordered set then the following conditions are equivalent :*

(1) E is a bounded lattice;

(2) there exists a balanced right Baer assembly $\langle A;B,D;K\rangle$ which coördinatises E.

Proof. (1)$\Rightarrow$(2): This is immediate from Theorem 1.

(2)$\Rightarrow$(1): Suppose that $\langle A;B,D;K\rangle$ is a balanced right Baer assembly which is such that $E \cong \mathcal{R}(D)$. For each $x \in D$ we have, by Theorem 8 of [1],

$$R_K(x) = (\hat{R}_K \circ \hat{L}_K)[R_K(x)] = (\hat{R}_K \circ \hat{L}_K)(x_* A) = \hat{R}_K[L_K(x)] = \hat{R}_K[A(x_*)^*] = R_K[(x_*)^*]$$

and, by the uniqueness in (1) and (2) of the definition,

$$R_K(x) = R_K(y) \iff x_* = y_* \iff (x_*)^* = (y_*)^*.$$

It follows that $\mathcal{R}(D) = \{R_K(x);\ x \in D\} = \{R_K(x);\ x \in D^*\}$. Moreover, $e \in B_*$ implies $e^* \in D^*$, so $\{R_K(e^*);\ e \in B_*\} \subseteq \{R_K(x);\ x \in D^*\}$; and $x \in D^* \iff x = (x_*)^* = e^*$ for some $e \in B_*$, so $\{R_K(x);\ x \in D^*\} \subseteq \{R_K(e^*);\ e \in B_*\}$. It follows that $\mathcal{R}(D) = \{R_K(x);\ x \in D^*\} = \{R_K(e^*);\ e \in B_*\}$. To show that, when ordered by set inclusion, $\mathcal{R}(D)$ is an $\wedge$-semilattice, it therefore suffices to show that, for $e,f \in B_*$, $R_K(e^*) \wedge R_K(f^*)$ is in $\mathcal{R}(D)$ whenever $R_K(e^*) \parallel R_K(f^*)$. Note that this is the same as the condition $eA \parallel fA$ since $R_K(e^*) = (e^*)_* A = eA$. Suppose then that $e,f \in B_*$ are such that $eA \parallel fA$. Then necessarily $e^*f \notin K$ and $f^*e \notin K$; for if, for example, $e^*f \in K$ then $f \in R_K(e^*) = eA$ and so $fA \subseteq eA$, a contradiction. Applying part (4) of the definition, we then have

$$(eA \parallel fA) \qquad eA \wedge fA = R_K(e^*) \wedge R_K(f^*) = R_K(e^*f)$$

and since $R_K(e^*f) = R_K[(e^*f)_*]^* \in \mathcal{R}(D)$ we see that $eA \wedge fA \in \mathcal{R}(D)$. This shows that $\mathcal{R}(D)$ forms an $\wedge$-semilattice. To show that $\mathcal{R}(D)$

also forms a union semilattice, suppose that $e, f \in B_*$ with $eA \parallel fA$. Then again $e^*f \notin K$ and $f^*e \notin K$ so, by (4), $L_K(e^*f) = L_K(e) \cap L_K(f)$ from which we have $L_K(e^*f) \subsetneq L_K(e)$ and $L_K(e^*f) \subsetneq L_K(f)$. It follows by Theorem 8 of [1] that $[(e^*f)^*]_*A = \hat{R}_K[L_K(e^*f)] \supsetneq \hat{R}_K[L_K(e)] = R_K(e^*)$ and similarly $\hat{R}_K[L_K(e^*f)] \supsetneq R_K(f^*)$. If now $h \in B_*$ is such that both $hA \supsetneq eA = R_K(e^*)$ and $hA \supsetneq fA = R_K(f^*)$ then $L_K(h) \subsetneq \hat{L}_K[R_K(e^*)] = Ae^*$ and similarly $L_K(h) \subsetneq Af^*$. It follows that $L_K(h) \subsetneq Ae^* \cap Af^* = L_K(e^*f)$ whence $hA = \hat{R}_K[L_K(h)] \supsetneq \hat{R}_K[L_K(e^*f)]$. This shows that

$$(eA \parallel fA) \qquad eA \vee fA = \hat{R}_K[L_K(e^*f)] = [(e^*f)^*]_*A = R_K[(e^*f)^*]$$

so that $\mathcal{R}(D)$ is a union semilattice in which unions are given by $\vee$ (which is in general different from set-theoretic union). We have thus shown that $\mathcal{R}(D)$ forms a bounded lattice, whence so also is E.

Remarks. (1) Although we shall not develop the details here, we point out that a parallel coördinatisation of bounded lattices can be obtained by using balanced *left* Baer assemblies <A;B,D;K>. In such an assembly we have $D \subsetneq B$ and $DB \subsetneq B$ and the "balancing" is obtained by changing part (3) of the previous definition to (3') $B_* \subsetneq D$. In this case the left Baer assembly used as a model is the left Baer assembly $\langle T;T,Res(E);\{0\}\rangle$ of §5 in [1].

(2) Although the definition of a balanced right Baer assembly is somewhat complicated, nevertheless the lattice operations are more readily represented in the coördinatising right Baer assembly than for instance was the case with the semilattice operation in Theorem 11 of [1]. Note that in the lattice case the mappings $\hat{\psi}_y$ belong to Res(E) whereas in the semilattice case in [1] the mappings ψ_y do not.

3. Coördinatisation of complemented lattices.

If $\langle A;B,D;K\rangle$ is a normal Baer assembly then, by Theorem 7 of [1], A has an identity element 1 with $1 \in B \cap D$. Also, K is generated by a central idempotent k° with $k^\circ \in B \cap D$ and k° is the identity element of K. If now $\langle A;B,D;K\rangle$ is a balanced right Baer assembly then from $1 \in B \cap D$ we have $K = R_K(1) = 1_*A$ and $K = L_K(1) = A1^*$. It follows that 1_* is a left identity for K and 1^* is a right identity for K, whence necessarily $1_* = 1^* = k^\circ$ and consequently $k^\circ \in B_* \cap D^*$. Similarly, from $k^\circ \in B \cap D$ we deduce that $A = R_K(k^\circ) = (k^\circ)_*A$ and $A = L_K(k^\circ) = A(k^\circ)^*$. These equalities show that $(k^\circ)_*$ is a left identity for A and $(k^\circ)^*$ is a right identity for A, whence we have $(k^\circ)_* = (k^\circ)^* = 1$ and consequently $1 \in B_* \cap D^*$. Note also that $K \cap B_* = \{k^\circ\} = K \cap D^*$; for example, if $k \in K \cap B_*$ then $A = L_K(k) = Ak^*$ so $1 = 1k^* = k^*$ and it follows that $k = (k^*)_* = 1_* = k^\circ$.

We can now isolate a balanced right Baer assembly which will coördinatise a complemented lattice.

THEOREM 3 *If E is an ordered set then the following conditions are equivalent :*

(1) E is a complemented lattice;

(2) E can be coördinatised by a balanced right Baer assembly $\langle A;B,D;K\rangle$ which satisfies the property

$$(C)\quad \begin{cases} (\forall e \in B_*\setminus\{k^\circ,1\})(\exists f \in B_*\setminus\{k^\circ,1\}) \\ R_K(e^*f) = R_K(f^*e) = L_K(e^*f) = L_K(f^*e) = K. \end{cases}$$

Proof. (1)$\Rightarrow$(2): Let E be a complemented lattice and consider the balanced right Baer assembly $\langle T;\mathrm{Res}(E),T\cap C;\{0\}\rangle$. Here we have

$k^\circ = 0$ and $1 = id_E$. If $0,\pi$ denote respectively the minimum and maximum elements of E then, as is readily seen, $\theta_x = 0 \iff x = 0$, $\theta_x = id_E \iff x = \pi$, $\hat{\psi}_x = 0 \iff x = \pi$ and $\hat{\psi}_x = id_E \iff x = 0$. It follows that, with $S = \mathrm{Res}(E)$,

$$\begin{cases} S_*\setminus\{0,id_E\} = \{\theta_x;\ x\notin\{0,\pi\}\}; \\ (T\cap C)^*\setminus\{0,id_E\} = \{\hat{\psi}_x;\ x\notin\{0,\pi\}\}. \end{cases}$$

Suppose then that $x\notin\{0,\pi\}$ and let x' denote any complement of x. We have

$$(\hat{\psi}_x \circ \theta_{x'})(y) = \begin{cases} \hat{\psi}_x(y) & \text{if } y \le x'; \\ \hat{\psi}_x(x') & \text{if } y \not\le x', \end{cases}$$
$$= \begin{cases} 0 & \text{if } y = 0; \\ x\cup y & \text{if } y \le x' \text{ and } y \not\le x; \\ \pi & \text{otherwise,} \end{cases}$$

from which it readily follows that $R(\hat{\psi}_x\circ\theta_{x'}) = \{0\} = L(\hat{\psi}_x\circ\theta_{x'})$. Interchanging x,x' gives similarly $R(\hat{\psi}_{x'}\circ\theta_x) = \{0\} = L(\hat{\psi}_{x'}\circ\theta_x)$ and the result follows.

(2)$\Rightarrow$(1): Suppose now that (2) holds and, given $e\in B_*\setminus\{k^\circ,1\}$, let $f\in B_*\setminus\{k^\circ,1\}$ be such that $R_K(e^*f) = R_K(f^*e) = L_K(e^*f) = L_K(f^*e) = K$. Then clearly $e^*f\notin K$ and $f^*e\notin K$ (for otherwise we would be reduced to the trivial case $K = A$) and hence $eA \,\|\, fA$. Applying the formulae of Theorem 2, we obtain

$$(eA \,\|\, fA) \qquad \begin{cases} eA\cap fA = R_K(e^*f) = K; \\ eA \vee fA = \hat{R}_K[L_K(e^*f)] = R_K(K) = A, \end{cases}$$

which shows that fA is a complement of eA in $\mathcal{R}(D)$. Observing that when $e = k^\circ$ we have $eA = k^\circ A = K$, which admits A as a complement in $\mathcal{R}(D)$, and that when $e = 1$ we have $eA = A$, which admits K as a complement in $\mathcal{R}(D)$, it follows that $\mathcal{R}(D)$ is a complemented lattice, whence so also is E.

Remark. For a coördinatisation of uniquely complemented lattices, it clearly suffices to postulate the uniqueness of f in part (2) of the above.

4. Coördinatisation of modular lattices.

In order to obtain a coördinatisation of bounded modular lattices, we require the assistance of other idempotents. By way of introducing these, we prove the following characterisation of modular lattices.

THEOREM 4 *Let E be a lattice and let U denote the semigroup of $\vee$-homomorphisms on E. Then the following conditions are equivalent:*

(1) E is modular;

(2) for all $a,b \in E$ with $a \| b$ the mapping $\theta_{a,b} : E \to E$ given by

$$\theta_{a,b}(x) = \begin{cases} a \wedge (b \vee x) & \text{if } x \nleq a \text{ and } x \leq a \vee b; \\ \theta_a(x) & \text{otherwise,} \end{cases}$$

is an element of U.

Proof. (1)$\Rightarrow$(2): Suppose that E is modular. Since clearly $\theta_a \in U$ for each $a \in E$ it suffices to show that $\theta_{a,b}(x \vee y) = \theta_{a,b}(x) \vee \theta_{a,b}(y)$ whenever $x \nleq a$, $x \leq a \vee b$ and y is any element of E. For this, we consider the following three cases:

(i) $x \nleq a$, $x \leq a \vee b$ *and* $y \leq a$:

In this case we have $\theta_{a,b}(x) = a \wedge (b \vee x)$ and $\theta_{a,b}(y) = \theta_a(y) = y$. Thus, since E is modular,

$$\theta_{a,b}(x) \vee \theta_{a,b}(y) = [a \wedge (b \vee x)] \vee y = a \wedge (b \vee x \vee y).$$

But from $x \nleq a$ we have $x \vee y \nleq a$; and since $y \leq a$ and $x \leq a \vee b$

we have $x \vee y \leq b \vee x \vee y \leq b \vee a \vee b \vee a = a \vee b$. Thus

$$\theta_{a,b}(x \vee y) = a \wedge (b \vee x \vee y) = \theta_{a,b}(x) \vee \theta_{a,b}(y).$$

(ii) $x \nleq a,\ x \leq a \vee b$ *and* $y \nleq a,\ y \leq a \vee b$:

In this case we have $\theta_{a,b}(x) = a \wedge (b \vee x)$ and $\theta_{a,b}(y) = a \wedge (b \vee y)$. The modularity of E then gives

$$\begin{aligned} \theta_{a,b}(x) \vee \theta_{a,b}(y) &= [a \wedge (b \vee x)] \vee [a \wedge (b \vee y)] \\ &= \{[a \wedge (b \vee x)] \vee b \vee y\} \wedge a && [\text{since } a \wedge (b \vee x) \leq a] \\ &= \{[(b \vee x) \wedge (a \vee b)] \vee y\} \wedge a && [\text{since } b \leq b \vee x] \\ &= (b \vee x \vee y) \wedge a && [\text{since } b \vee x \leq a \vee b]. \end{aligned}$$

But from $x \nleq a$ we have $x \vee y \nleq a$; and since both $x, y \leq a \vee b$ we have $x \vee y \leq a \vee b$. Hence

$$\theta_{a,b}(x \vee y) = a \wedge (b \vee x \vee y) = \theta_{a,b}(x) \vee \theta_{a,b}(y).$$

(iii) $x \nleq a,\ x \leq a \vee b$ *and* $y \nleq a,\ y \nleq a \vee b$:

In this case we have $\theta_{a,b}(x) = a \wedge (b \vee x)$ and $\theta_{a,b}(y) = \theta_a(y) = a$ so that

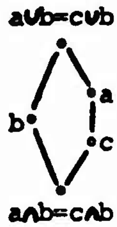
$$\theta_{a,b}(x) \vee \theta_{a,b}(y) = [a \wedge (b \vee x)] \vee a = a.$$

But since also $x \vee y \nleq a$ and $x \vee y \nleq a \vee b$ we see that

$$\theta_{a,b}(x \vee y) = \theta_a(x \vee y) = a = \theta_{a,b}(x) \vee \theta_{a,b}(y).$$

Parts (i), (ii), (iii) now show that, whenever $a \parallel b$, we have $\theta_{a,b} \in U$ as required.

(2)$\Rightarrow$(1): By way of obtaining a contradiction, suppose that (2) holds and that E is not modular. As is well known, E then contains a sublattice of the form



Now in this sublattice we observe that

$$\begin{cases} \theta_{a,b}(b\vee c) = a\wedge(b\vee b\vee c) = a\wedge(a\vee b) = a\ ; \\ \theta_{a,b}(b)\vee\theta_{a,b}(c) = [a\wedge(b\vee b)]\vee c = c, \end{cases}$$

so that $\theta_{a,b}\notin U$, a contradiction. This then establishes (2)$\Rightarrow$(1).

Remark. Observe that if we remove the restriction that $a\|b$ in the definition of $\theta_{a,b}$ then we obtain $\theta_{a,b} = \theta_a$. For if $b \leq a$ then $x \leq a\vee b$ implies $x \leq a$, so $\{x\in E;\ \theta_{a,b}(x) = a\wedge(b\vee x)\} = \emptyset$; and if $b > a$ then $x \nleq a$, $x \leq a\vee b = b$ imply $\theta_{a,b}(x) = a\wedge(b\vee x) = a\wedge b = a = \theta_a(x)$. Thus we need distinguish $\theta_{a,b}$ only in the case where $a\|b$.

The principal properties of the mappings $\theta_{a,b}$ when $a\|b$ in a bounded modular lattice are listed in the following result.

THEOREM 5 *Let E be a bounded modular lattice and let T be the semigroup of quasi-residuated $\vee$-homomorphisms on E. If $a,b\in E$ are such that $a\|b$ then*

(1) $\theta_{a,b}$ *is an idempotent of* T;

(2) $R(\hat{\psi}_a) = \theta_a\circ T = \theta_{a,b}\circ T$;

(3) $L(\theta_{a,b}) = L(\theta_a) = T\circ\hat{\psi}_a$;

(4) $L(\theta_{a,b}\circ\theta_b) = \hat{L}[R(\hat{\psi}_a\circ\theta_b)] = L(\theta_{a\wedge b})$;

(5) $L(\hat{\psi}_a\circ\theta_{a,b}) = L(\hat{\psi}_b\circ\theta_a) = L(\theta_{a\vee b})$.

Proof. (1) Theorem 4 shows that $\theta_{a,b}$ is a $\vee$-homomorphism and is consequently isotone. To show that $\theta_{a,b}\in T$ it then suffices to note that $\theta_{a,b}(0) = \theta_a(0) = 0$. To show that $\theta_{a,b}$ is idempotent, it suffices, by virtue of the fact that θ_a is idempotent, to show that whenever $x \not\leq a$ and $x \leq a \vee b$ we have $\theta_{a,b}(x) = \theta_{a,b}[\theta_{a,b}(x)]$. Now for such an element x we have

$$\theta_{a,b}[\theta_{a,b}(x)] = \theta_{a,b}[a \wedge (b \vee x)] = \theta_a[a \wedge (b \vee x)] = a \wedge (b \vee x) = \theta_{a,b}(x).$$

(2) In the proof of Theorem 1 we observed that $R(\hat{\psi}_a) = \theta_a \circ T$. To establish the equality $\theta_{a,b} \circ T = \theta_a \circ T$ it is sufficient, by (1), to show that $\theta_{a,b} = \theta_a \circ \theta_{a,b}$ and $\theta_a = \theta_{a,b} \circ \theta_a$. Now $\theta_a(x) \leq a$ for all $x \in E$ and when $y \leq a$ we have $\theta_{a,b}(y) = \theta_a(y)$. It follows that $\theta_{a,b}[\theta_a(x)] = \theta_a[\theta_a(x)] = \theta_a(x)$ for each $x \in E$ and so $\theta_{a,b} \circ \theta_a = \theta_a$. Also, $\theta_{a,b}(x) \leq a$ for each $x \in E$ and so we have $\theta_a[\theta_{a,b}(x)] = \theta_{a,b}(x)$, which shows that $\theta_a \circ \theta_{a,b} = \theta_{a,b}$.

(3) This is immediate from (2) on taking left annihilators.

(4) As observed in the proof of Theorem 1, we have

$$\alpha \in R(\hat{\psi}_a \circ \theta_b) \iff \operatorname{Im} \alpha \subseteq [\leftarrow, a \wedge b] \iff \alpha \in R(\hat{\psi}_{a \wedge b}).$$

It follows from this that $R(\hat{\psi}_a \circ \theta_b) = R(\hat{\psi}_{a \wedge b}) = \theta_{a \wedge b} \circ T$. Taking left annihilators we obtain $\hat{L}[R(\hat{\psi}_a \circ \theta_b)] = L(\theta_{a \wedge b})$. Now

$$(\theta_{a,b} \circ \theta_b)(x) = \begin{cases} \theta_{a,b}(x) \leq \theta_{a,b}(b) = a \wedge b & \text{if } x \leq b; \\ \theta_{a,b}(b) = a \wedge b & \text{if } x \not\leq b, \end{cases}$$

and so we deduce that

$$\alpha \in L(\theta_{a,b} \circ \theta_b) \iff \alpha(a \wedge b) = 0 \iff \alpha \in L(\theta_{a \wedge b}),$$

which completes the proof of (4).

(5) As observed in the proof of Theorem 1, we have

$$\alpha \in L(\hat{\psi}_b \circ \theta_a) \iff \alpha(a \vee b) = 0 \iff \alpha \in L(\theta_{a \vee b})$$

and so $L(\hat{\psi}_b \circ \theta_a) = L(\theta_{a \vee b})$. Now

$$(\hat{\psi}_b \circ \theta_{a,b})(x) = \begin{cases} \hat{\psi}_b[a \wedge (b \vee x)] \leq \hat{\psi}_b(a) = a \vee b & \text{if } x \nleq a,\ x \leq a \\ \hat{\psi}_b[\theta_a(x)] = \hat{\psi}_b(x) = x \vee b \leq a \vee b & \text{otherwise,} \end{cases}$$

with $(\hat{\psi}_b \circ \theta_{a,b})(a) = a \vee b$. It follows that

$$\alpha \in L(\hat{\psi}_a \circ \theta_{a,b}) \iff \alpha(a \vee b) = 0 \iff \alpha \in L(\theta_{a \vee b}),$$

which completes the proof of (5).

We are now in a position to provide a coördinatisation of bounded modular lattices.

THEOREM 6 *If* E *is an ordered set then the following conditions are equivalent:*

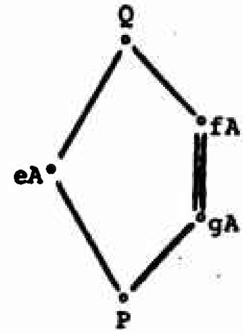
(1) E *is a bounded modular lattice;*

(2) E *can be coördinatised by a balanced right Baer assembly* $\langle A;B,D;K\rangle$ *which satisfies the property*

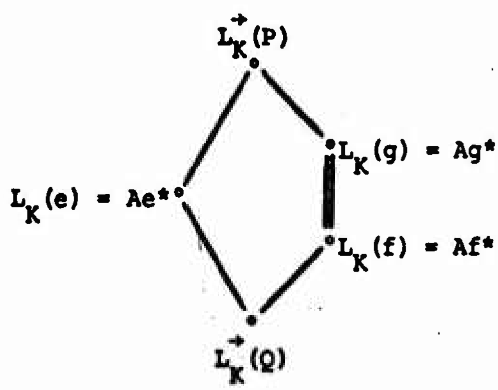
$$(M) \quad \begin{cases} \text{if } e,f \in B_* \text{ are such that } e^*f \notin K \text{ and } f^*e \notin K \\ \text{then there exists } \bar{e} = \bar{e}^2 \in A \text{ with } \bar{e}A = eA \text{ and} \\ R_K(e^*f) = \hat{R}_K[L_K(\bar{e}f)]. \end{cases}$$

Proof. (1)$\Rightarrow$(2): Let E be a bounded modular lattice. To establish (2), it suffices to show that $\langle T;\mathrm{Res}(E),T\cap C;\{0\}\rangle$ satisfies (M). Now given $\theta_a,\theta_b \in [\mathrm{Res}(E)]_*$ such that $\hat{\psi}_a \circ \theta_b \neq 0$ and $\hat{\psi}_b \circ \theta_a \neq 0$ we have as before $a \| b$ and $\theta_{a,b} \in T$ with $\theta_{a,b}$ idempotent. Moreover, by Theorem 5, $\theta_{a,b} \circ T = \theta_a \circ T$. Taking right annihilators in Theorem 5(4), we obtain finally $\hat{R}[L(\theta_{a,b} \circ \theta_b)] = R(\hat{\psi}_a \circ \theta_b)$. This then shows that the condition (M) is satisfied.

(2)$\Rightarrow$(1): Suppose now that (2) holds. To show that E is a modular lattice it suffices to show that, if $\mathcal{R}(D)$ has a sublattice of the form



in which increasing single lines denote $\subset$, the increasing double line denotes $\subsetneq$, $Q = eA \vee fA = eA \vee gA$ and $P = eA \wedge fA = eA \wedge gA$, then necessarily $fA = gA$. Now since the mapping $\vec{L}_K : \mathcal{R}(D) \to \mathcal{L}(B)$ which is defined by $\vec{L}_K(eA) = \hat{L}_K(eA) = L_K(e) = Ae^*$ is a dual isomorphism (see [1], Theorem 8), it is equivalent to show that if $\mathcal{L}(B)$ has a sublattice of the form



then necessarily $Af^* = Ag^*$. Using the formulae of Theorem 2, we see that $Q = eA \vee gA = \hat{R}_K[L_K(e^*g)]$ and, by the hypothesis (2), that $P = fA \wedge eA = R_K(f^*e) = \hat{R}_K[L_K(\bar{f}e)]$. It follows that $\vec{L}_K(P) = L_K(\bar{f}e)$ and $\vec{L}_K(Q) = L_K(e^*g)$. The problem therefore reduces to showing that

$$L_K(f) \supsetneq L_K(g) = L_K(g) \wedge \vec{L}_K(P),$$

which is equivalent to

$$Af^* \supsetneq Ag^* = Ag^* \wedge L_K(\bar{f}e).$$

Now if $x \in Ag^* \cap L_K(\bar{f}e)$ then on the one hand $x = xg^*$ and on the other $x\bar{f}e \in K$. The second of these gives $x\bar{f} \in L_K(e) = Ae^*$ and so $x\bar{f} = x\bar{f}e^*$. But $gA \subsetneq fA = \bar{f}A$ and so $g = \bar{f}g$ whence $xg = x\bar{f}g = x\bar{f}e^*g$. Since $x = xg^*$ we then have $xg = xg^*g \in K$ and hence $x\bar{f}e^*g \in K$. It follows that $x\bar{f} \in L_K(e^*g) = \vec{L}_K(Q) \subsetneq L_K(f)$ and so $x\bar{f}f \in K$. But since $\bar{f}A = fA$ we have $\bar{f}f = f$. We thus deduce that $xf \in K$ and so $x \in L_K(f) = Af^*$ as required. This then establishes that $\mathcal{R}(D)$, and hence E, is a bounded modular lattice.

Remark. Note that in the condition (M) the element $\bar{e}$ is simply an element of A and not necessarily in B or D. However, since $\vec{L}_K$ is a dual isomorphism and $\hat{R}_K[L_K(\bar{e}f)] \in \mathcal{R}(D)$, we see that $L_K(\bar{e}f) = \vec{L}_K\hat{R}_K[L_K(\bar{e}f)] \in \mathcal{L}(B)$.

5. Coördinatisation of distributive lattices.

In the case of a distributive lattice, the idempotents which we require are of a much more obvious nature. These are given in the following result, the proof of which is clear.

THEOREM 7 *Let E be a lattice and for each $x \in E$ let $t_x : E \to E$ be given by $t_x(y) = x \wedge y$. Then each t_x is idempotent and, when E is bounded, the following conditions are equivalent :*

(1) E is distributive;

(2) $(\forall x \in E)\quad t_x \in T$.

The next result shows how the idempotents t_x enjoy, in a bounded distributive lattice, properties similar to those listed above for the mappings $\theta_{a,b}$ in a bounded modular lattice.

THEOREM 8 *Let E be a bounded distributive lattice and let T be the semigroup of quasi-residuated $\vee$-homomorphisms on E. Then given $a,b \in E$ we have*

(1) $L(t_a) = L(\theta_a)$; $t_a \circ T = \theta_a \circ T$;

(2) $L(t_a \circ \theta_b) = L(\theta_{a\wedge b}) = L(t_a \circ t_b)$;

(3) $L(\hat{\psi}_a \circ t_b) = L(\theta_{a\vee b})$.

Proof. (1) Since for each $\alpha \in T$ we have $(\alpha \circ t_a)(x) = \alpha(a\wedge x) \leq \alpha(a) = (\alpha \circ t_a)(a)$, we see that $\alpha \circ t_a = 0$ if and only if $\alpha(a) = 0$, so that $L(t_a) = L(\theta_a)$. That $t_a \circ T = \theta_a \circ T$ follows from the facts that both t_a and θ_a are idempotent,

$$t_a[\theta_a(x)] = a\wedge\theta_a(x) = \begin{cases} a\wedge x = x & \text{if } x \leq a; \\ a\wedge a = a & \text{if } x \not\leq a, \end{cases}$$
$$= \theta_a(x),$$

and $\theta_a[t_a(x)] = \theta_a(a\wedge x) = a\wedge x = t_a(x)$.

(2) Since

$$(t_a \circ \theta_b)(x) = \begin{cases} a\wedge x \leq a\wedge b & \text{if } x \leq b; \\ a\wedge b & \text{if } x \not\leq b, \end{cases}$$

we see that, in the semigroup T,

$$\alpha \in L(t_a \circ \theta_b) \iff \alpha(a\wedge b) = 0 \iff \alpha \in L(\theta_{a\wedge b}).$$

Clearly, these are also equivalent to $\alpha \in L(t_a \circ t_b)$.

(3) Since

$$(\hat{\psi}_a \circ t_b)(x) = \hat{\psi}_a(b\wedge x) = \begin{cases} 0 & \text{if } a\wedge x \leq b; \\ (b\wedge x)\vee a \leq b\vee a & \text{if } a\wedge x \not\leq b, \end{cases}$$

with $(\hat{\psi}_a \circ t_a)(\pi) = a\vee b$, we see that

$$\alpha \in L(\hat{\psi}_a \circ t_b) \iff \alpha(a\vee b) = 0 \iff \alpha \in L(\theta_{a\vee b}).$$

We can now provide a coördinatisation of bounded distributive lattices.

THEOREM 9 *If* E *is an ordered set then the following conditions are equivalent :*

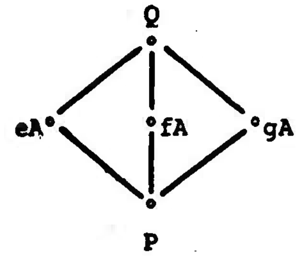
(1) E *is a bounded distributive lattice;*

(2) E *can be coördinatised by a balanced right Baer assembly* $\langle A;B,D;K\rangle$ *which satisfies the property*

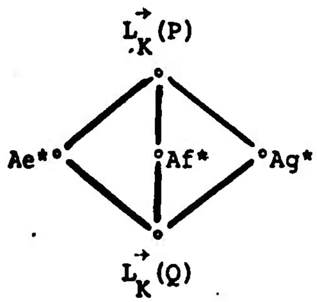
$$(D)\left\{\begin{array}{l}\text{there exists an abelian idempotent subsemigroup } \hat{B}_* \\ \text{of } A \text{ such that} \\ \quad \text{(i) for each } e\in B_* \text{ there is a unique } \bar{e}\in\hat{B}_* \\ \text{such that } \bar{e}A = eA; \\ \quad \text{(ii) if } e,f\in B_* \text{ are such that } e^*f\notin K \text{ and } f^*e\notin K \\ \text{then } R_K(e^*f) = \hat{R}_K[L_K(\bar{e}f)] = \hat{R}_K[L_K(\bar{e}\bar{f})].\end{array}\right.$$

Proof. (1)$\Rightarrow$(2): Suppose that (1) holds. Then it suffices to show that $\langle T;\mathrm{Res}(E),T\cap C;\{0\}\rangle$ satisfies the condition (D). To this end we observe first that $\{t_a;\ a\in E\}$ clearly forms an abelian idempotent subsemigroup of T. Writing $S = \mathrm{Res}(E)$, define $\hat{S}_* = \{t_a;\ a\in E\}$. We see that $\hat{S}_*$ satisfies (i) since, by Theorem 8(1), $\theta_a \circ T = t_a \circ T$ and if $t_a \circ T = t_b \circ T$ then clearly $t_a = t_b$. As for (ii), we note that if $\theta_a,\theta_b\in S_*$ are such that $\hat{\psi}_a \circ \theta_b \neq 0$ and $\hat{\psi}_b \circ \theta_a \neq 0$ then $a \| b$ and we see from Theorem 8(2) that $R(\hat{\psi}_a \circ \theta_b) = \hat{R}[L(t_a \circ \theta_b)] = \hat{R}[L(t_a \circ t_b)]$.

(2)$\Rightarrow$(1): Suppose that (2) holds. From the condition (D) we see that $\langle A;B,D;K\rangle$ satisfies the condition (M) of Theorem 6 and so $\mathcal{R}(D)$ is a bounded modular lattice. To show that it is distributive, it therefore suffices to show that it contains no sublattice of the form



As in the proof of Theorem 6, we can apply the dual isomorphism $\vec{L}_K$ and show equivalently that $\mathcal{L}(B)$ contains no sublattice of the form



To fix ideas, we shall assume that Q, P, eA and fA are related as is shown above, so that $Q = eA \vee fA = \hat{R}_K[L_K(e^*f)]$ and, by (D), $P = eA \wedge fA = R_K(e^*f) = \hat{R}_K[L_K(\bar{e}\bar{f})]$ where $\bar{e},\bar{f} \in \hat{B}_*$. Then $\vec{L}_K(Q) = L_K(e^*f)$ and $\vec{L}_K(P) = L_K(\bar{e}\bar{f})$. We shall now show that the conditions

$$L_K(e^*f) \subsetneq Ag^* \subsetneq L_K(\bar{e}\bar{f}), \quad Ag^* \wedge Af^* = \vec{L}_K(Q) \quad \text{and} \quad Ag^* \parallel Af^*$$

together imply that $Ag^* \subsetneq Ae^*$ from which it will follow that the above sublattice degenerates and $\mathcal{R}(D)$ is distributive. This we do as follows.

We note first that for $\bar{g} \in \hat{B}_*$ we have $R_K(g^*) = gA = \bar{g}A$ and so $g^*\bar{g} \in K$. Since $\bar{e}\bar{g} = \bar{g}\bar{e}$ we then have $g^*\bar{e}\bar{g} = g^*\bar{g}\bar{e} \in K$ and hence

$$g^*\bar{e} \in L_K(\bar{g}) = \hat{L}_K(\bar{g}A) = \hat{L}_K(gA) = Ag^*,$$

from which it follows that $g^*\bar{e} = g^*\bar{e}g^*$. Since $Ag^* \subsetneq \vec{L}_K(P) = L_K(\bar{e}\bar{f})$ we have $g^*\bar{e}\bar{f} \in K$. Using the previous equality, we deduce that $g^*\bar{e}g^*\bar{f} \in K$ so that $g^* \in L_K(\bar{e}g^*\bar{f})$ and consequently $Ag^* \subsetneq L_K(\bar{e}g^*\bar{f})$.

Now using the readily established fact that

$$L_K(x) \subseteq L_K(y) \Rightarrow (\forall z \in A) \quad L_K(zx) \subseteq L_K(zy)$$

we see from the fact that $L_K(f) = L_K(\bar{f})$ that

$$L_K(\bar{e}g^*\bar{f}) = L_K(\bar{e}g^*f).$$

Similarly, from $L_K(\bar{e}) \supseteq \vec{L}_K(Q) = L_K(g^*f)$ we obtain

$$Ae^* = L_K(e) = L_K(\bar{e}) = L_K(\bar{e}\bar{e}) \supseteq L_K(\bar{e}g^*f).$$

Combining the above observations, we deduce that $Ae^* \supseteq Ag^*$ as required.

Remarks. (1) In the above proof it was not necessary to show that $L_K(\bar{e}g^*f) \in \mathcal{L}(B)$. That this is so follows from the fact that a similar proof with e,g interchanged yields $Ae^* = Ag^* = L_K(\bar{e}g^*\bar{f}) = L_K(\bar{g}e^*\bar{f})$.

(2) In [1] we showed that a bounded ordered set E forms an implicative (Glivenko-Brouwer) semilattice if and only if E can be coördinatised by a right Baer assembly $\langle A;B,D;K\rangle$ in which there is an abelian idempotent subsemigroup $\hat{B}$ of B such that $\mathcal{R}(D) = \{eA;\ e \in \hat{B}\}$. Since a lattice E is implicative if and only if $t_a \in \mathrm{Res}(E)$ for each $a \in E$, it is easy to see from the preceding result that a distributive lattice is implicative if and only if it can be coördinatised by a balanced right Baer assembly which (using the notation introduced above) satisfies the additional condition that $B_* = \hat{B}_*$ or, equivalently, $\hat{B}_* \subseteq B$. This illustrates once again that the essential difference between the lattice and the semilattice coördinatisations is the "balancing factor" of §2.

REFERENCE

[1] T.S. BLYTH and W.C. HARDY : Quasi-residuated mappings and Baer assemblies, *Proc.Roy.Soc.Edinburgh (to appear).*

List of CNA Professional Papers*

PP 1
Brown, George F. and Lloyd, Richmond M., "Static Models of Bank Credit Expansion," 27 pp., 23 Sep 1969, (Published in the Journal of Financial and Quantitative Analysis, Jun 1971) AD 703 925

PP 2
Lando, Mordechai E., "The Sex-Differential in Canadian Unemployment Data," 5 pp., 9 Jan 1970, AD 699 512

PP 3
Brown, George F.; Corcoran, Timothy M. and Lloyd, Richmond M., "A Dynamic Inventory Model with Delivery Lag and Repair," 16 pp., 1 Aug 1969, AD 699 513

PP 4
Kadane, Joseph B., "A Moment Problem for Order Statistics," 14 pp., 13 Jan 1970, (Published in the Annals of Mathematical Statistics, Apr 1971) AD 699 514

PP 5
Kadane, Joseph B., "Optimal Whereabouts Search," 28 pp., Oct 1969, (Published in the Journal of the Operations Research Society of America, Vol. XIX, 1971) AD 699 515

PP 6 – Classified

PP 7
Friedheim, Robert L., "The Continental Shelf Issue at the United Nations: A Quantitative Content Analysis," 25 pp., 7 Jan 1970, (To be published in "Pacem in Maribus," edited by Elaine H. Burnell and Piers von Simson, Center for the Study of Democratic Instructions) (See also PP 28) AD 699 516

PP 8
Rose, Marshall and White, Alex, "A Comparison of the Importance of Economic Versus Non-Economic Factors Affecting the Residential Housing Market During the Two Decades Subsequent to World War II," 128 pp., 15 Jan 1970, AD 699 517

PP 9
Rose, Marshall, "A Thesis Concerning the Existence of Excess Capacity at Naval Shipyards Prior to the Escalation of Hostilities in Southeast Asia in 1964," 67 pp., 9 Jan 1970, AD 699 518

PP 10 – Classified

PP 11
O'Neill, David M., "The Effect of Discrimination on Earnings: Evidence from Military Test Score Results," 19 pp., 3 Feb 1970, (Published in the Journal of Human Resources, Summer 1970) AD 703 926

PP 12
Brown, George F. and Lloyd, Richmond M., "Dynamic Models of Bank Credit Expansion Under Certainty," 29 pp., 3 Feb 1970, AD 703 931

PP 13
Overholt, John L., "Analysis Data Inputs and Sensitivity Tests in War Games," 30 pp., Mar 1971, AD 722 858

PP 14
Rose, Marshall, "Determination of the Optimal Investment in End Products and Repair Resources," 38 pp., 18 Feb 1970, (Published in the Annual Meeting of the American Association of Cost Engineers Proceedings, Jun 1971, Montreal, Canada) AD 702 450

*CNA Professional Papers with an AD number may be obtained from the National Technical Information Service, U.S. Department of Commerce, Springfield, Virginia 22151. Other papers are available from the author at the Center for Naval Analyses, 1401 Wilson Boulevard, Arlington, Virginia 22209.

PP 15
Rose, Marshall, "Computing the Expected-End Product Service Time Using Extreme Value Properties of Sampling Distribution," 29 pp., 18 Feb 1970, (Published in Operations Research, Mar-Apr 1971) AD 702 451

PP 16
Rose, Marshall, "Study of Reparable Item Resupply Activities," 35 pp., 18 Feb 1970, AD 702 452

PP 17
Brown, Lee (Lt., USN) and Rose, Marshall, "An Incremental Production for the End-Item Repair Process," 17 pp., 3 Mar 1970, (Published in Annual Conference of the American Institute of Industrial Engineers Transactions, May 1970, Cleveland, Ohio) AD 702 453

PP 18
Rose, Marshall, "Inventory and the Theory of the Firm," 14 pp., 18 Feb 1970, AD 702 454

PP 19
Rose, Marshall, "A Decomposed Network Computation for End-Product Repair Curves," 24 pp., 18 Feb 1970, AD 702 455

PP 20
Brown, George F.; Corcoran, Timothy M. and Lloyd, Richmond M., "Inventory Models with a Type of Dependent Demand and Forecasting, with an Application to Repair," 4 pp., 10 Feb 1970, (Published in Management Science: Theory Section, Mar 1971) AD 702 456

PP 21
Silverman, Lester P., "Resource Allocation in a Sequential Flow Process," 21 pp., 5 Mar 1970, AD 702 457

PP 22
Gorlin, Jacques, "Israeli Reprisal Policy and the Limits of U.S. Influence," 27 pp., 23 Mar 1970, AD 703 534

PP 23
Rose, Marshall, "An Aircraft Rework Cost-Benefit Model," 13 pp., 12 Mar 1970, (Published in the 5th Annual DoD Cost Research Symposium Proceedings, Mar 1970) AD 702 514

PP 24
Lloyd, Richmond and Sutton, S. Scott, "An Application of Network Analysis to the Determination of Minimum Cost Aircraft Pipeline Factors," 51 pp., 31 Mar 1970, (Presented at NATO Conference on Problems in the Organization and Introduction of Large Logistic Support Systems, May 1970, Luxembourg) AD 703 536

PP 25
Saperstone, Stephen, "An Approach to Semi-Markov Processes," 38 pp., 23 Mar 1970, AD 703 537

PP 26
Brown, George F. and Corcoran, Timothy M., "The Reliability of a Complex System with Spares, Repair, and Cannibalization," 45 pp., 23 Mar 1970, AD 703 538

PP 27
Fain, Janice B.; Fain, William W.; Feldman, Leon and Simon, Susan, "Validation of Combat Models Against Historical Data," 18 pp., 14 Apr 1970, (Published in 9th Symposium of the National Gaming Council Proceedings, Apr 1970) AD 704 744

PP 28
Friedheim, Robert L. and Kadane, Joseph B., "Quantitative Content Analysis of the United Nations Seabed Debates: Methodology and a Continental Shelf Case Study," 32 pp., 24 Mar 1970, (Published in International Organization, Vol. XXIV, No. 3, 1970) AD 703 539

PP 29
Saperstone, Stephen H., "Controllability of Linear Oscillatory Systems Using Positive Controls," 27 pp., Apr 1970, AD 704 745

PP 30
DeVany, Arthur S., "The Effect of Social Policy on the Social and Private Value of a Child," 20 pp., 27 Apr 1970, AD 704 746

PP 31
DeVany, Arthur S., "Time in the Budget of the Consumer," 51 pp., 27 Apr 1970, AD 704 747

PP 32
Overholt, John L., "Fitting Korean War Data by Statistical Method," 11 pp., 5 May 1970, (Presented at the 9th Symposium of the National Gaming Council, Apr 1970, Washington, D.C.) AD 705 349

PP 33
DeVany, Arthur S., "A Theory of Household Demand and Labor Supply," 23 pp., 5 May 1970, AD 705 350

PP 34
Kadane, Joseph B. and Fisher, Franklin M.*, "The Covariance Matrix of the Limited Information Estimator and the Identification Test: Comment," 6 pp., 14 May 1970, (To be published in Econometrica) AD 706 310
*Department of Economics, Massachusetts Institute of Technology

PP 35
Lando, Mordechai E., "Full Employment and the New Economics–A Comment," 4 pp., 14 May 1970, (Published in the Scottish Journal of Political Economy, Vol. XVII, Feb 1969) AD 706 420

PP 36
DeVany, Arthur S., "Time in the Budget of the Consumer: The Theory of Consumer Demand and Labor Supply Under a Time Constraint," 151 pp., 15 Jun 1970, AD 708 348

PP 37
Kadane, Joseph B., "Testing a Subset of the Overidentifying Restrictions," 7 pp., 19 Jun 1970, (To be published in Econometrica) AD 708 349

PP 38
Saperstone, Stephen H., "The Eigenvectors of a Real Symmetric Matrix are Asymptotically Stable for Some Differential Equation," 19 pp., Jul 1970, AD 708 502

PP 39
Hardy, W. C. and Blyth, T. S.*, "Quasi-Residuated Mappings and Baer Assemblies," 22 pp., 14 Jul 1970, (To be published by the Royal Society of Edinburgh)
*Mathematical Institute, University of St. Andrew

PP 40
Silverman, Lester P. and Forst, Brian E., "Evaluating Changes in the Health Care Delivery System: An Application to Intensive Care Monitoring," 19 pp., Jul 1970, AD 710 631

PP 41
Piersall, Charles H. (LCdr), "An Analysis of Crisis Decision-Making," 55 pp., Aug 1970, (To be published in the American Political Science Review) AD 719 705

PP 42
Sullivan, John A., "Measured Mental Ability, Service School Achievement and Job Performance," 22 pp., 31 Jul 1970, AD 720 359

PP 43
Forst, Brian E., "Estimating Utility Functions Using Preferences Revealed under Uncertainty," 13 pp., Jun 1971, (Presented at the 39th National Meeting of the Operations Research Society of America, 5 May 1971) AD 726 472

PP 44
Schick, Jack M., "Conflict and Integration in the Near East: Regionalism and the Study of Crises," 43 pp., Oct 1970, (Presented at the 66th Annual Meeting of the American Political Science Association, Sep 1970)

PP 45
Brown, George F. and Lloyd, Richmond M., "Fixed Shortage Costs and the Classical Inventory Model," 13 pp., Jul 1970, AD 713 057

PP 46
Hardy, William C. and Blyth, T. S.*, "A Coordinatisation of Lattices by One-Sided Baer Assemblies," 21 pp., Jul 1970, (To be published by the Royal Society of Edinburgh)
*Mathematical Institute, University of St. Andrew

PP 47
Silverman, Lester, P., "Resource Allocation in a Sequential Flow Process with an Application to the Naval Resupply System," 18 pp., Oct 1970, (Presented at the 11th American Meeting of the Institute of Management Sciences, Oct 1970; Presented at the 26th Meeting of the Military Operations Research Society, Nov 1970) AD 713 028

PP 48
Gray, Burton C., "Writeup for B34TCNA–A Step-Wise Multiple Regression Program," 15 pp., Oct 1970, AD 713 029

PP 49
Friedheim, Robert L., "International Organizations and the Uses of the Ocean," 88 pp., Oct 1970, (To be published in Volume of Essays on International Administration, Oxford University Press AD 714 387

PP 50
Friedheim, Robert L. and Kadane, Joseph B., "Ocean Science in the United Nations Political Arena," 45 pp., Jun 1971, AD 731 865

PP 51
Saperstone, Stephen H., "Global Controllability of Linear Systems with Positive Controls," 29 pp., Nov 1970, AD 714 650

PP 52
Forst, Brian E., "A Decision-Theoretic Approach to Medical Diagnosis and Treatment," 14 pp., Nov 1970, (Presented at the Fall 1970 11th American Meeting of the Institute of Management Sciences, Oct 1970, Los Angeles, California) AD 714 651

PP 53
Kadane, Joseph B., "On Division of the Question," 12 pp., Nov 1970, (Published in Public Choice, Fall 1971) AD 714 652

PP 54
Kadane, Joseph B., "How to Burgle If You Must: A Decision Problem," 13 pp., May 1971, AD 723 850

PP 55
Brown, George F., "Optimal Management of Bank Reserves," 35 pp., Aug 1970, AD 715 569

PP 56
Horowitz, Stanley A., "Economic Principles of Liability and Financial Responsibility for Oil Pollution," 26 pp., Mar 1971, AD 722 376

PP 57
Lando, Mordechai E., "A Comparison of the Military and Civilian Health Systems," 20 pp., Dec 1970, AD 716 897

PP 58
Kadane, Joseph B., "Chronological Ordering of Archeological Deposits by the Minimum Path Length Method," 16 pp., Jun 1971, AD 726 475

PP 59
Dyckman, Zachary Y., "An Analysis of Negro Employment in the Building Trades," 309 pp., Jan 1971

PP 60
Lando, Mordechai E., "Health Services in the All Volunteer Armed Force," 33 pp., Jan 1971, (Published in Studies Prepared for the President's Commission on an All-Volunteer Force, Government Printing Office, Nov 1970) AD 716 899

PP 61
Robinson, Jack, "Classification Management Training and Operations, An Approach," 14 pp., Jul 1971, (Presented at the 7th Annual Seminar, National Classification Management Society, Washington, D.C., 13-16 Jul 1971) AD 727 719

PP 62
Brown, George F. and Schwartz, Arnold N., "The Cost of Squadron Operation: A Theoretical and Empirical Investigation," 10 pp., Jan 1971 (Published in the Transactions of the 1971 American Association of Cost Engineers International Meeting, Jun 1971) AD 722 377

PP 63
Lockman, Robert F., "Analyses of Selection and Performance Measures for CNA Support Personnel," 45 pp., Feb 1971, AD 720 360

PP 64
Utgoff, Victor A. and Kashyap, R L.*, "On Behavior Strategy Solutions in Two-Person Zero-Sum Finite Extended Games with Imperfect Information," 37 pp., Feb 1971, (Accepted for publication in the SIAM Journal on Applied Mathematics) AD 720 361
*School of Electrical Engineering, Purdue University

PP 65
O'Neill, David M.; Gray, Burton C. and Horowitz, Stanley, "Educational Equality and Expenditure Equalization Orders: The Case of Hobson V. Hansen," 43 pp., Feb 1971, AD 720 362

PP 66
Schwartz, Arnold N.; Sheler, James A. (LCdr) and Cooper, Carl R. (Cdr), "Dynamic Programming Approach to the Optimization of Naval Aircraft Rework and Replacement Policies," 39 pp., Mar 1971, (To be published in the Naval Research Logistics Quarterly) AD 720 363

PP 67
Kuzmack, Richard A., "Measures of the Potential Loss from Oil Pollution," 16 pp., Mar 1971, (Published as Chapter 13 in Legal, Economic, and Technical Aspects of Liability and Financial Responsibility as Related to Oil Pollution, The George Washington University, Dec 1970) AD 722 378

PP 68
Blechman, Barry M. and Holt, James, T., "Cost/Effectiveness Analysis of Foreign Policy Alternatives: Need, Approach, and Prospects," 41 pp., Mar 1971, (Presented at the 1971 Western Regional Meeting of the International Studies Association, Mar 1971) AD 722 379

PP 69
Rogers, Warren F. (Cdr), "Exact Null Distributions of Rank Test Statistics," 47 pp., Mar 1971, AD 722 380

PP 70
Rogers, Warren F. (Cdr), "On A Theorem of Weyl," 17 pp., Mar 1971, AD 722 381

PP 71
Lloyd, Richmond M., "Dynamic Programming Models of Short Term Bank Reserve Management," 233 pp., Mar 1971, AD 727 724

PP 72
Kadane, Joseph B. and Iversen, Gudmund R.*, "Estimation of Multinomial Process When Only the Sum and the Number Governed by Each Process is Observed," 13 pp., Apr 1971, AD 722 382
*University of Michigan

PP 73
Victor A. Utgoff and Kashyap, R. L.*, "On Behavior Strategy Solutions in Two-Person Zero-Sum Finite Extended Games with Imperfect Information, Part II: Determination of a Minimally Complex Behavior Strategy Solution in a Medical Decision Process," 22 pp., May 1971, (Accept for publication in the SIAM Journal on Applied Mathematics) AD 723 851
*School of Electrical Engineering, Purdue University

PP 74
Brown, Jr. George F.; Silverman, Lester P. and Perlman, Bernard L. (AWF3), "Optimal Positioning of Inventory Stock in a Multi-Echelon System," 37 pp., May 1971, (Presented at the 39th Annual Meeting of the Operations Research Society of America, May 1971) AD 723 852

PP 75
Stoloff, Peter H., "The Navy Personal Response Program: Review, Evaluation and Recommendations," 22 pp., Jul 1971, AD 727 725

CNA Professional Papers – Cont'd.

PP 76
Canes, Michael E., "Measurement and Selection of Defense," 21 pp., Aug 1971, AD 731 868

PP 77
McConnell, James M., "The Soviet Navy in the Indian Ocean," 16 pp., Aug 1971, AD 731 869

PP 78
Blechman, Barry M., "A Quantitative Description of Arab-Israeli Interactions, 1949-1969: Data Sets and Processor," 43 pp., Sep 1971, AD 731 870

PP 79
Wilson, Desmond P. and Brown, Nicholas (Cdr), "Warfare at Sea: Threat of the Seventies," 14 pp., Nov 1971, AD 734 856

PP 80
Weinland, Robert G., "The Changing Mission Structure of the Soviet Navy," 15 pp., Nov 1971, AD 734 077

PP 81
Forst, Brian, E., "The Grisly Analytics of Death, Disability, and Disbursements," 20 pp., Nov 1971, (Presented at the 40th National Meeting of the Operations Research Society of America, 28 Oct 71) AD 732 555

PP 82
Forst, Brian E., "A Doctor's Introduction to Decision Analysis," 22 pp., Nov 1971, (Presented at the Engineering Foundation Conference on Quantitative Decision Making for the Delivery of Ambulatory Care, 22 Jul 1971) AD 732 556

PP 83
Weiher, Rodney and Horowitz, Stanley A., "The Relative Costs of Formal and On-the-Job Training for Navy Enlisted Occupations," 44 pp., Nov 1971, AD 734 857

PP 84
Weiher, Rodney and Horowitz, Stanley A., "A Production Function for Trained Recruits," 27 pp., Nov 1971, AD 734 858

PP 85
Brown, Jr. George F., "Comparison of Forecast Accuracy When the Disturbances Are Small: Directly Estimated Reduced Forms vs. K-Class Induced Reduced Forms," 17 pp., Dec 1971, AD 736 355

PP 86
Harrison, Robert A., "Multivariate Regression Analysis and Slaughter Livestock," 33 pp., Dec 1971, AD 736 356

PP 87
Harkins, James A., "Computer Software: A Major Weapon System Component," 9 pp., Jan 1972, (Presented at the Washington Chapter ACM 10th Annual Symposium, 24 Jun 71, National Bureau of Standards, Gaithersburg, Md.) AD 736 357

PP 88
Lockman, Robert F., "An Evaluation of the Applicant Interview Form for CNA Support Jobs," 19 pp., Mar 1972

PP 89
Barney, Gerald O., "System Dynamics and the Analysis of Social Policy," 27 pp., Apr 1972, (Presented at the XIX International Meeting of the Institute of Management Sciences, Apr 4-8, 1972, Houston, Tex.)

PP 90
Heider, Dr. Charles H., "An N-Step, 2-Variable Search Algorithm for the Component Placement Problem," 58 pp., Apr 1972

PP 91
Piersall, Jr. Charles H. and Borgstrom, Robert E., "Cost Analysis of Optional Methods of Shipboard Domestic Waste Disposal," 23 pp., Apr 1972, (To be presented at the Annual Northeast Regional Anti-pollution Conference at the University of Rhode Island, Jul 1972)

PP 92
Forst, Brian E., "Decision Analysis and Medical Malpractice," 30 pp., May 1972, (Presented at the 41st National Meeting of the Operations Research Society of American, New Orleans, La., 27 Apr 1972)

PP 93

Zedlewski, Edwin W., "Estimation and Inference in Binary Response Regressions," 52 pp., May 1972